Canonical quantization of non-commutative holonomies in 2+1 loop quantum gravity

K. Noui
Laboratoire de Mathématiques et Physique Théorique, Parc de Grammont, Tours, France.

A. Perez, D. Pranzetti
Centre de Physique Théorique,
Campus de Luminy, Marseille, France.

In this work we investigate the canonical quantization of 2+1 gravity with cosmological constant $\Lambda > 0$ in the canonical framework of loop quantum gravity. The unconstrained phase space of gravity in 2+1 dimensions is coordinatized by an $SU(2)$ connection A and the canonically conjugate triad field e. A natural regularization of the constraints of 2+1 gravity can be defined in terms of the holonomies of $A_\pm = A \pm \sqrt{\Lambda}e$. As a first step towards the quantization of these constraints we study the canonical quantization of the holonomy of the connection $A_\lambda = A + \lambda e$ (for $\lambda \in \mathbb{R}$) on the kinematical Hilbert space of loop quantum gravity. The holonomy operator associated to a given path acts non trivially on spin network links that are transversal to the path (a crossing). We provide an explicit construction of the quantum holonomy operator. In particular, we exhibit a close relationship between the action of the quantum holonomy at a crossing and Kauffman's q-deformed crossing identity (with $q = \exp(i\hbar\lambda/2)$). The crucial difference is that (being an operator acting on the kinematical Hilbert space of LQG) the result is completely described in terms of standard $SU(2)$ spin network states (in contrast to q-deformed spin networks in Kauffman's identity). We discuss the possible implications of our result.

I. INTRODUCTION

The link between the Jones Polynomial, Chern-Simons theory and quantum gravity in 2+1 dimensions with non vanishing cosmological constant has been first shown by Witten in the seminal papers [1]. First, he showed that 2+1 dimensional (first order) gravity can be reformulated in terms of a Chern-Simons theory whose gauge algebra is the isometry algebra of the local solutions of Einstein equations. Then, he proposed a path integral quantization of the Chern-Simons theory with compact gauge Lie groups G. In the case where $G = SU(2)$, this quantization is closely related to the quantization of Euclidean gravity with a positive cosmological constant, which is the only situation where the gauge group is compact. The work of Witten has opened an incredible rich new way of understanding 3-manifolds and knots invariants because the expectation values of Wilson loops observables in Chern-Simons theory has lead to a new covariant definition of the Jones polynomials and its generalizations.

After this result, it was precisely shown by Reshetikhin and Turaev [2] that quantum groups play a central role in the construction of 3-manifolds invariants and knots polynomials. The construction of the Turaev-Viro invariant is a very nice illustration of this fact [6]. These invariants can be viewed as a q-deformed version of Ponzano and Regge amplitudes. Moreover, the asymptotic of the vertex amplitudes (the quantum $6j$-symbol) has been shown to be related to the action of $2+1$ gravity with non vanishing cosmological constant in the WKB approximation [7].

All this, strongly motivates the idea that it should be possible to recover (in the context of loop quantum gravity [11]) the Turaev-Viro amplitudes as the physical transition amplitudes of 2+1 gravity with non-vanishing cosmological constant. This has been so far explicitly shown only in the simpler case for pure gravity with vanishing cosmological constant [8].

Can we find a clear-cut relationship between the Turaev-Viro amplitudes and the transition amplitudes computed from the canonical quantization of 2+1 gravity with non vanishing cosmological constant? Using the so-called combinatorial quantization, developped in the compact case in [3] and then generalized in non-compact situations in [4] and [5], one shows how quantum groups appear in the canonical quantization and therefore one makes a link between covariant and canonical quantizations of gravity. However, quantum groups do not appear in this framework from a bottom-up approach but they are putten by hand for purposes of regularization. The kinematical Hilbert space is finite dimensional and expressed already in terms of quantum groups. Physical states are obtained solving the quantum constraints that reduce, in that case, to requiring invariance under the quantum group adjoint action. The combinatorial quantization is certainly one of the most powerful canonical quantization of 2+1 dimensional gravity because it is, to our knowledge, the only quantization scheme that leads to an explicit construction of the physical Hilbert space for any topology of the space surface.

Loop quantum gravity in 2+1 dimensions is another framework where it is possible to address this question. The advantage of working with loop quantum gravity instead of with the combinatorial quantization is that it could help us understanding quantum gravity in four dimensions. As in the combinatorial quantization, we starts by quantizing the unreduced phase space of the theory and then imposes the constraints at the quantum level (Dirac recipe). But, contrary to the combinatorial quantization (where the non-reduced phase space is finite dimensional), there is an infinite number of degrees of freedom before imposing the constraints, which in the case of $2+1$ gravity are encoded in the infinitely many polymer-like excitations represented by spin network states. In LQG it is natural to interpret the Turaev-Viro invariant as transition amplitudes between arbitrary pairs of such graph-based states. Now, if the previous statement makes sense, the Turaev-Viro amplitudes would have to be related to the kinematical states of the canonical theory, namely classical $SU(2)$ spin networks. In contrast the Turaev-Viro amplitudes are constructed from the combinatorics of q-deformed spin networks [16]. This would imply that the understanding of the relationship between the Turaev-Viro invariants and quantum gravity requires the understanding of the dynamical interplay between classical spin-network states and q-deformed amplitudes. We shall find here some indications about how this relationship can arise.

Let us first briefly recall the canonical structure of (Riemannian) gravity in 2+1 dimensions.

The action of departure is

$$S(A,e) = \int_{\mathcal{M}} \mathrm{tr}\left[e \wedge F\left(A\right)\right] + \frac{\Lambda}{6}\mathrm{tr}\left[e \wedge e \wedge e\right] \ ,$$

where $\Lambda \geq 0$, e is a cotriad field, and A is an $SU(2)$ connection.

Assuming that the space time manifold has topology $\mathcal{M} = \Sigma \times \mathbb{R}$, and, upon the standard 2+1 decomposition, the phase space of the theory is parametrized by the pullback to Σ of ω and e. In local coordinates we can express them in terms of the 2-dimensional connection A_a^i and the triad field e_a^i where $a = 1, 2$ are space coordinate indices and $i, j = 1, 2, 3$ are $su(2)$ indices. The Poisson bracket among these is given by

$$\{A_a^i\left(x\right), e_b^j\left(y\right)\} = \epsilon_{ab}\ \delta_j^i \delta^{(2)}\left(x,\, y\right) \tag{1}$$

where ϵ_{ab} is the 2d Levi-Civita tensor. The phase space variables are subjected to the first class local constraints

$$d_A e = 0 \quad \text{ and } \quad F(A) + \Lambda\ e \wedge e = 0 \tag{2}$$

The basic kinematical observables are given by the holonomy of the connection and appropriately smeared functionals of the triad field e. Quantization of these (unconstrained) observables leads to an irreducible representation on a Hilbert space, the so-called kinematical Hilbert space $\mathscr{H}_k$, with a diffeomorphism invariant inner product (see [12] and references therein): states in $\mathscr{H}_k$ are given by functionals $\Psi[A]$ of the (generalized) connection A which are square-integrable with respect to a diff-invariant measure. The holonomy acts simply by multiplication while e acts as the derivative operator $e_a^i = -\mathrm{i}\hbar\epsilon_{ab}\ \delta_j^i \delta/\delta A_b^j$ (more precisely, the objects that correspond to the field e in loop quantum gravity are the flux operators associated to curves in Σ, see Section III).

Dynamics is defined by imposing the quantum constraints (defined by the representation of (2) as self adjoint operators in $\mathscr{H}_k$) on the kinematical states. More precisely, the quantum constraint-equations of 2+1 gravity with cosmological constant can be written as

$$\mathcal{G}\left[\alpha\right] \triangleright \Psi = \int_{\Sigma} \mathrm{Tr}[\alpha \mathrm{d}_A e] \triangleright \Psi = 0 \tag{3}$$

and

$$C_\Lambda\left[N\right] \triangleright \Psi = \int_{\Sigma} \mathrm{Tr}\left[N\left(F(A) + \Lambda\, e \wedge e\right)\right] \triangleright \Psi = 0 \tag{4}$$

for all $\alpha, N \in \mathcal{C}^\infty(\Sigma, su(2))$. The previous equations are formal at this stage. The difficulty resides in the fact that the constraints are non linear functional of the basic fields and their quantization requires the introduction of a regularization. Therefore, the precise meaning of the previous equations is a subtle issue which will be at least partially investigated in this work.

In [8] the quantization and solution of the equations above for the special case $\Lambda = 0$ is completely worked out. More precisely, the construction of the physical Hilbert space of 2+1 gravity is achieved by means of a rigorous implementation of the Dirac quantization program to the theory. A natural result of this work is the definition of the path integral representation of the theory from the canonical picture. This establishes the precise relationship between the physical inner product of 2+1 gravity and the spin foam amplitudes of the Ponzano-Regge model[1]. In addition to providing a systematic definition of the quantum theory, the canonical treatment has the advantage of automatically avoiding the infrared divergences that plagued Ponzano-Regges original construction. Another advantage of the formulation is that it sets the bases for the extension of

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[1] See [9] for a more recent and alternative investigation of the link between the canonical quantization of the Wheeler-DeWitt equation and the symmetries of the Ponzano-Regge model.

the analysis to the non vanishing cosmological constant case[2]. Indeed, the key observation is that equation (4) can be quantized by first introducing a regulator consisting of a cellular decomposition Δ_Σ of Σ—with plaquettes $p \in \Delta_\Sigma$ of coordinate area smaller or equal to ϵ^2—so that

$$C_0\,(N) = \int_\Sigma \mathrm{Tr}\,[N\,F\,(A)] = \lim_{\epsilon\to 0} \sum_{p\in\Delta_\Sigma} \mathrm{Tr}\,[N_p\,W_p\,(A)]\ , \tag{5}$$

where $W_p(A) = 1 + \epsilon^2 F(A) + o(\epsilon^2) \in SU(2)$ is the Wilson loop computed in the fundamental representation. The quantization of the previous expression is straightforward as the Wilson loop acts simply by multiplication on the kinematical states of 2+1 gravity. Then, the Ponzano-Regge

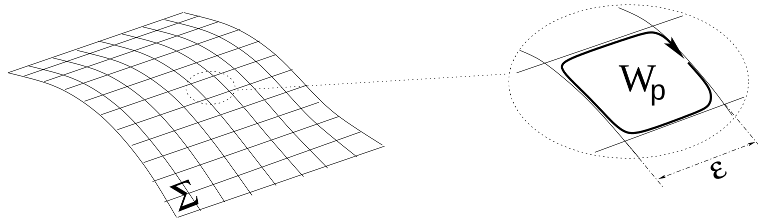


Figure 1. Cellular decomposition of the space manifold Σ (a square lattice in this example), and the infinitesimal plaquette holonomy $W_p[A]$.

amplitudes can be recovered through the definition of a physical scalar product by means of a projector operator into the kernel of (5). A key ingredient for this construction turns out to be, together with the background independence of the whole approach, the absence of anomaly in the quantum algebra of the constraints. In the case of $\Lambda \neq 0$, this is no longer the case, as shown in [20] (see [21] for a possible way around this difficulty).

Here, we propose an alternative approach to the problem of 2+1 gravity with $\Lambda \neq 0$ in the context of LQG. We start from the observation that, if we replace $W_p(A)$ by $W_p(A_\pm)$ (with $A_\pm = A \pm \sqrt{\Lambda} e$) on the previous equation, a simple calculation shows that at the classical level we get

$$C_\Lambda\,[N] = \lim_{\epsilon\to 0} \sum_{p\in\Delta_\Sigma} \mathrm{Tr}\,[N_p\,W_p\,(A_\pm)] - \mathcal{G}\left[\pm\sqrt{\Lambda} N\right]. \tag{6}$$

This provides a candidate background independent regularization of the curvature constraint $C_\Lambda[N]$ for arbitrary values of the cosmological constant. Notice that on gauge invariant states (i.e. the solution space of the Gauss constraint) the second term simply drops out. The quantization of the previous classical expression requires the quantization of the holonomy of $A_\pm$. More generally, as a first step towards the quantization of (6), in the present work we study the quantization of the holonomy h_λ of the general connection $A_\lambda \equiv A + \lambda e$ for $\lambda \in \mathbb{R}$. The difficulties in the quantization of h_λ arise from the fact that it is a *non-commutative* holonomy, since function of a connection (A_λ) becomes itself non-commutative upon quantization, as clear from the Poisson bracket (1).

The paper is organized as follows: In Sections II we give a brief account of our results avoiding technical details. In Section III we briefly recall the quantization scheme of the e-field in the LQG formalism. In Section IV the technical results are exhibit in detail. The crossing between quantum holonomies is defined in terms of a series expansion in powers of the cosmological constant. We prove that the series is well defined and can be summed to produce a simple result. However, the result depends on quantization choices. The choice of some natural prescription, such as the fully symmetrized ordering, yields unsatisfactory results, as shown in Section V A. In Section V B we

[2] For a pedagogical review on the link between the physical inner product and spin foams see [17]. For more general basic literature about spin foams see [18]. Recent results on the connection between LQG and spin foams in 4d can be found in [19].

briefly introduce the Duflo isomorphism which provides a preferred quantization map in a given sense. In Section V C, we compute the action of the quantum holonomy defined by a suitable implementation of the Duflo map in the LQG formalism. The action of an quantum holonomy on a transversal holonomy (both in the fundamental representation) exactly reproduces Kauffmann's bracket. In Section VI, we discuss the possible implications of our results in the framework of the question raised in this introduction. Some technical material is presented in the Appendices.

II. THE RESULTS IN A NUT-SHELL

In this work we explore the quantization of the (one parameter family of) classical (kinematical) observables

$$h_\eta\left[A_\lambda\right] = P\ \mathrm{e}^{-\int_\eta A+\lambda e} \tag{7}$$

associated with a path $\eta \in \Sigma$, as operators on the kinematical Hilbert space of 2+1 loop quantum gravity.

Due to the tensorial form of the Poisson bracket (1) (inherited by the commutator in the quantum theory) the action of (7) on the vacuum simply creates a Wilson line excitation, i.e. it acts simply by multiplication by the holonomy of A along the path, namely

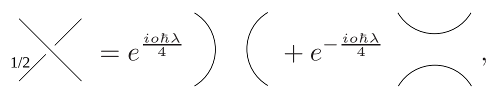
$$h_\eta\left[A_\lambda\right] \triangleright 1 = h_\eta\left[A\right]. \tag{8}$$

This is because the e-operator in the argument of the path ordered exponential in (7) acts as a derivative operator with respect to the components of the connection that are transversal to the curve (notice the presence of the ϵ_{ab} in the canonical commutation relations (1)). The action of the holonomy of A_λ is therefore expected not to be trivial when the loop α in (7) is self intersecting or when it acts on generic spin-network states containing vertices on (or edges transversal to) α.

Therefore, the simplest non-trivial example is the action on a transversal Wilson loop in the fundamental representation. We define the quantization of (7) by quantizing each term in the series expansion of (7) in powers of λ. Terms of order n have n powers of the e operators. The quantization of these products becomes potentially ill-defined due to factor ordering ambiguities (operators associated to e are non commuting in the quantum theory [10]).

The same kind of problem has been recently investigated in [24], where the authors provided an new derivation of the expectation values of holonomies in Chern-Simons theory. In the analysis of [24], the same sort of ordering ambiguities arises due to the replacement of holonomy functionals under the path integral with a complicated functional differential operator; the authors show that the expected result can be recovered once a mathematically preferred ordering, dictated by the Duflo isomorphism, is adopted[3]. Therefore, following the example of [24], we will also make use of this mathematical insight, but in our case the Duflo map will not do the all job. In fact, since the ambiguities in the quantization of (7) arise due to the presence of non-linear terms in the e-field, a second piece of information has to be taken into account, namely the quantum action of flux operators in LQG. Combining these two elements leads to a well defined quantization for each term in the perturbative expansion in λ. Moreover, the series can be summed and the result can be expressed in a closed form, leading to algebraic structures remarkably equal to those appearing in Kauffman's q-deformed spin networks.

More precisely, if we concentrate on a single intersection (a *crossing*) between the path defining the holonomy of A_λ and a transversal spin-network edge in the fundamental representation $j = 1/2$ we obtain



(9)

[3] For another application of the Duflo map in the context of 2 + 1 quantum gravity see also [23].

where o is the orientation of the crossing. Therefore, even though the crossing of paths happens on the two dimensional manifold Σ, a distinction between over and under crossing on the lhs of the previous expression is still possible according to the relative orientations of the path on which the quantum holonomy is defined and of the spin network edge it acts on. The action (9) reproduce exactly Kaufmann's q-deformed crossing identity, where the deformation parameter reads $q = A^2 = e^{\frac{i\hbar\lambda}{2}}$.

Despite of the strict resemblance of the previous equation and the Kauffman bracket, there objects appearing in equation (9) are quite diffrerent from the ones in Kauffman's identity. Here, the paths involved are elements of $\mathscr{H}_k$ of LQG, i.e. classical $SU(2)$ holonomies. For that reason the famous Reidemeister identity as well as the Yang-Baxter braid identity that can be derived from the analog of (9) in the knot theory context are not valid here. Equation (9) are a different kind of quantum deformation of the Maldestam relation for $SU(2)$ (the binor spinorial identity) that we find using canonical quantization of (7). This is the a central result of our work.

The fact that our crossing does not satisfy the topological properties of strands in knot theory deserves more qualification. As it is well known 2+1 gravity is a topological theory with no local degrees of freedom. In the computation of expectation values of knotted (spacetime-embeded) Wilson loops, this implies that their value is a knot-invariant as it is shown in [1]. From the viewpoint of the canonical loop quantum gravity canonical approach (where one reduces after quantization) this is expected to hold only on shell, i.e., after having imposed the quantum constraint (4). Our quantization of (7) is constructed at the level of the kinematical Hilbert space where there are infinitely many local (pure gauge) degrees of freedom. At that level there is no *a priori* reason for the crossing to be topological. We will further discuss this point in Section VI.

III. QUANTIZATION OF e-FIELD

In LQG there is a well-defined quantization of the e-field based on the smearing of e along one dimensional paths. More precisely, given a path $\eta^a(t) \in \Sigma$ one considers the quantity

$$E(\eta) \equiv \int e^i_a \tau_i \frac{d\eta^a}{dt} dt = \int E^{ai} \tau_i n_a dt, \tag{10}$$

where in the second equation we have replaced e in terms of the connection conjugate momentum E^a_i and $n_a \equiv \epsilon_{ab} \frac{d\eta^a}{dt}$ is the normal to the path. Therefore, the previous quantity represents the flux of E across the curve η. The quantum operator associated to $E(\eta)$ acts non trivially only on holonomies h_γ along a path $\gamma \in \Sigma$ that are transversal to η. It sufices to give its action on trasnversal holonomies that either end or start on η. The result is:

$$\hat{E}(\eta) \triangleright h_\gamma = \frac{1}{2}\hbar \begin{cases} o(p)\tau^i \otimes \tau_i h_\gamma & \text{if } \gamma \text{ ends at } \eta \\ o(p) h_\gamma \tau^i \otimes \tau_i & \text{if } \gamma \text{ starts at } \eta \end{cases}, \tag{11}$$

where $o(p)$ is the orientation of the intersection $p \in \Sigma$ (denoted p for puncture), namely

$$o(p) = \left. \frac{\epsilon_{ab}\dot{\eta}^a \dot{\gamma}^b}{|\epsilon_{ab}\dot{\eta}^a \dot{\gamma}^b|} \right|_p \tag{12}$$

at the intersection $p \in \Sigma$. In other words the operator $E(\eta)$ acts at a puncture as an $SU(2)$ left-invariant-vector-field (LIV) if the puncture is the source of h_γ, and it acts as a right-invariant-vector-field (RIV) if the puncture is the target of h_γ. This observation will lead to a natural regularization of the quantum holonomy operator (7) is what follows.

IV. QUANTIZATION OF $h(A_\lambda)$

Let $\Sigma \times \mathbb{R}$ be a global decomposition of the $2+1$ dimensional spacetime, $\gamma, \eta : (0,1) \to \Sigma$ two curves that cross each other *transversally* in $\gamma(s_*) = \eta(t_*)$. Let $A = A^i_a \mathrm{d}x^a \otimes \tau_i$ be a connection on

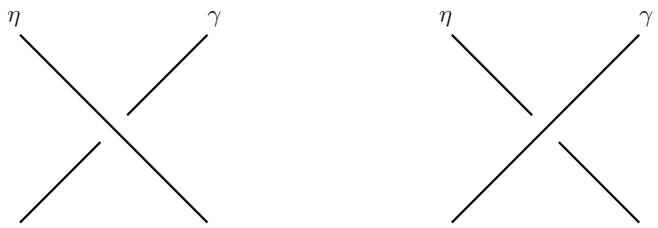


Figure 2. Graphical representation of the action of two quantum holonomies $h_\eta(A_\lambda)$ and $h_\gamma(A_\lambda)$. The three dimensional structure depicted as over-crossing or under crossing encodes operator ordering. In this way the picture on the left denotes the operator action $h_\eta(A_\lambda) \triangleright h_\gamma(A_\lambda)$ while the one on the right denotes $h_\gamma(A_\lambda) \triangleright h_\eta(A_\lambda)$.

a principal $SU\,(2)$-bundle over $\Sigma \times \mathbb{R}$, for which we choose a trivialization around $\gamma\,(s_*) = \eta\,(t_*)$. Let $h_\gamma\,(A)$ denote the holonomy of γ in this trivialization. Let $(A_\lambda)^i_a = A^i_a + \lambda\, e^i_a = A^i_a + \lambda\,\epsilon_{ab} E^b_i$, E^b_i being the momentum canonically conjugate to A^i_a.

Let us show that the action of $h_\eta[A_\lambda]$ on the vacuum is trivial, namely

$$h_\eta[A_\lambda]|0\rangle = h_\eta[A]|0\rangle, \tag{13}$$

which is simply equivalent to equation (8) were we use Dirac's bracket-notation for the vacuum whose wave functional $\langle A|0\rangle = 1$. The momenta E^b_i are formally quantized as $E^b_i\,(x) \mapsto -\mathrm{i}\hbar\delta/\delta A^b_i\,(x)$. In order to give a meaning to the quantum operator $h_\eta\,(A_\lambda)$ we first develop its classical expression in powers of λ and obtain, for the generic pth order,

$$\lambda^p \sum_{n\geq p}\sum_{m\geq p} (-1)^{m+n} \sum_{1\leq k_1<\cdots<k_p\leq n} \int_0^1 \mathrm{d}t_1 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n \int_0^1 \mathrm{d}s_1 \cdots \int_0^{s_{m-1}} \mathrm{d}s_m$$
$$\left[A\,(\eta\,(t_1)) \cdots A\,(\eta\,(t_{k_1-1}))\; E(\eta(t_{k_1})) \cdots E(\eta(t_{k_p}))\, A\left(\eta\left(t_{k_p+1}\right)\right) \cdots A\,(\eta\,(t_n))\right]\,|0\rangle\;.$$

As the commutator

$$\left[E\,(\eta\,(t_k))\;, A\,(\eta\,(t_p))\right] = 0, \tag{14}$$

due to the fact that both fields in the commutator are pulled-back on the same curve, only the $p = 0$ term of the previous series survives when acting on the vacuum. Thus (13) follows. The previous argument is formal: choosing a system of coordinates $(s,\,t)$ around η (which we suppose sufficiently small) in which η be represented by $\eta\,(t) = (0,\,t)$ we see that $\delta\,(\eta\,(t_p) - \eta\,(t_k)) = \delta\,((0,\,t_p) - (0,\,t_k)) = \delta\,(0)\,\delta\,(t_p - t_k)$ is singular. Nevertheless, a more careful treatment based on a suitable regularization where the flux line is replaced by a flux tube (defined by a smooth thickening of the path η) leads to the same conclusion [15] as our formal shortcut.

Let us move on now and study of the action of η on γ. Denoting this action by "$\triangleright$" and using the previous results, we have:

$$h_\eta\,(A_\lambda) \triangleright h_\gamma\,(A_\lambda)\,|0\rangle = h_\eta\,(A_\lambda) \triangleright h_\gamma\,(A)\,|0\rangle =$$
$$\left(1 + \sum_{1\leq n}(-1)^n \int_0^1 \mathrm{d}t_1 \cdot\cdot \int_0^{t_{n-1}} \mathrm{d}t_n\, A_\lambda\,(\eta\,(t_1)) \cdot\cdot A_\lambda\,(\eta\,(t_n))\right) \triangleright$$
$$\left(1 + \sum_{1\leq m}(-1)^m \int_0^1 \mathrm{d}s_1 \cdot\cdot \int_0^{s_{m-1}} \mathrm{d}s_m\, A\,(\gamma\,(s_1)) \cdot\cdot A\,(\gamma\,(s_m))\right) |0\rangle\,.$$

Developing in powers of λ the coefficient at order p is:

$$\lambda^p \sum_{n\geq p}\sum_{m\geq p}(-1)^{m+n}\sum_{1\leq k_1<\cdots<k_p\leq n}\int_0^1 \mathrm{d}t_1\cdots\int_0^{t_{n-1}}\mathrm{d}t_n\int_0^1 \mathrm{d}s_1\cdots\int_0^{s_{m-1}}\mathrm{d}s_m$$
$$\left[A\left(\eta\left(t_1\right)\right)\cdots E(\eta(t_{k_1}))\cdots E(\eta(t_{k_p}))\cdots A\left(\eta\left(t_n\right)\right)\right]\triangleright A\left(\gamma\left(s_1\right)\right)\cdots A\left(\gamma\left(s_m\right)\right)\,.$$

In what follows we shall omit the sums $\sum_{n\geq p}$, $\sum_{m\geq p}$ and the coefficient $(-1)^{m+n}$, and we shall only restore them at the end of the calculations. Let us concentrate on the action of the derivation operators on the connection along γ. The relevant quantity is

$$\int_0^1 \mathrm{d}s_1\cdots\int_0^{s_{m-1}}\mathrm{d}s_m\; E(\eta(t_{k_1}))\cdots E(\eta(t_{k_p}))\triangleright A\left(\gamma\left(s_1\right)\right)\cdots A\left(\gamma\left(s_m\right)\right)\,. \tag{15}$$

One now uses

$$\begin{aligned} E(\eta(t))\triangleright A\left(\gamma\left(s\right)\right) &= \left(\epsilon_{ab}\dot{\gamma}^a\left(s_*\right)\dot{\eta}^b\left(t^*\right)\right)\delta\left(\gamma\left(s\right)-\eta\left(t\right)\right)\\ &= \delta\left(s-s_*\right)\delta\left(t-t_*\right)\frac{\epsilon_{ab}\dot{\gamma}^a\left(s_*\right)\dot{\eta}^b\left(t_*\right)}{\left|\epsilon_{ab}\dot{\gamma}^a\left(s_*\right)\dot{\eta}^b\left(t_*\right)\right|}\\ &= o\;\delta\left(s-s_*\right)\delta\left(t-t_*\right), \end{aligned} \tag{16}$$

where o is the orientation of the intersection defined by taking γ and η in this order [4]. It is easy to see that only the terms containing p consecutive graspings $E(\eta(t_q)), E(\eta(t_{q+1}))$ up to $E(\eta(q_{q+p}))$ which themselves act on p consecutive $A(\gamma(s_k)), A(\gamma(s_{k+1}))$ up to $A(\gamma(s_{k+p}))$ survive. Any other possible term will vanish as a consequence of the previous equation (the domain of integration of the integrals of A's evaluated on intermediate parameters will be constrained to a single point by the delta functions (16)). The Leibnitz rule now produces a sum over all possible orderings for the action of the E on the sequence $A(\gamma(s_k)), A(\gamma(s_{k+1}))$ up to $A(\gamma(s_{k+p}))$. Finally, a factor $(1/p!)^2$ is produced by the ordered integral of p two dimensional delta distributions[5]. One can arrange the integration variables and get

$$\begin{aligned} &\frac{(-\mathrm{i}o\hbar\lambda)^p}{p!}\sum_{k_1\geq 1}(-1)^{k_1-1}\int_{t_*}^1 \mathrm{d}t_1\cdots\int_{t_*}^{t_{k_1-2}}\mathrm{d}t_{k_1-1}\,A\left(\eta\left(t_1\right)\right)\cdots A\left(\eta\left(t_{k_1-1}\right)\right)\\ &\tau^{i_{k_1}}\cdots\tau^{i_{k_p}}\sum_{v\geq 0}(-1)^v\int_0^{t_*}\mathrm{d}\tilde{t}_1\cdots\int_0^{t_{v-1}}\mathrm{d}\tilde{t}_v\,A\left(\eta\left(\tilde{t}_1\right)\right)\cdots A\left(\eta\left(\tilde{t}_v\right)\right)\otimes\\ &\sum_{\alpha_{k_1}\geq 1}(-1)^{\alpha_{k_1}-1}\int_{s_*}^1 \mathrm{d}s_1\cdots\int_{s_*}^{s_{\alpha_{k_1}-2}}\mathrm{d}s_{\alpha_{k_1}-1}\,A\left(\gamma\left(s_1\right)\right)\cdots A\left(\gamma\left(s_{\alpha_{k_1}-1}\right)\right)\\ &\tau_{(i_{k_1}}\cdots\tau_{i_{k_p})}\sum_{u\geq 0}(-1)^u\int_0^{s_*}\mathrm{d}\tilde{s}_1\cdots\int_0^{s_{u-1}}\mathrm{d}\tilde{s}_u\,A\left(\gamma\left(\tilde{s}_1\right)\right)\cdots A\left(\gamma\left(\tilde{s}_u\right)\right)\,, \end{aligned} \tag{17}$$

where in the last line the brackets on the subindexes denote symmetrization, namely

$$\tau_{(i_1}\cdots\tau_{i_p)} = \frac{1}{p!}\sum_{\pi\in S(p)}\tau_{i_{\pi(1)}}\cdots\tau_{i_{\pi(p)}}, \tag{18}$$

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[4] There is an additional relative minus sign between under and over crossing. This can entirely encoded in o if we choose the paths ordered according to the operator action (see Figure 2) and its caption.

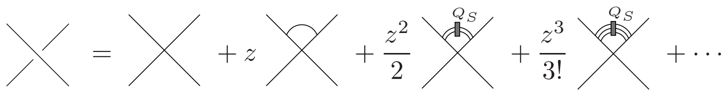
[5] Here we are using that
$$\int_K \delta(t_1)\cdots\delta(t_n)\,F(t_1,\cdots,t_n) = \frac{1}{p!}F(0,\cdots,0),$$
where $K = \{t = (t_1,\cdots,t_p)\in\mathbb{R}^p\,|-\infty < t_p\leq\cdots\leq t_1<\infty\}$.

for $S(p)$ denoting the group of permutations of p. The insertion of the symmetrized product of generators can be thought of as the action of a quantization prescription defined by the map

$$\mathrm{Q}_S: \; E_{i_1} E_{i_2} \cdots E_{i_p} \quad \rightarrow \quad \frac{1}{p!} \sum_{\pi \in S(p)} \tau_{i_{\pi(1)}} \tau_{i_{\pi(2)}} \cdots \tau_{i_{\pi(p)}}. \tag{19}$$

As we have shown in the manipulations of this section, the previous quantization map arises naturally from the Leibniz rule in our context. There are however factor ordering ambiguities due to the non-commutativity of the grasping operators that allow in principle for other prescriptions (that we will call Q in the following section). We will see in what follows that the advertised relationship with the Kauffman bracket is found if one uses the so-called Duflo map instead.

For further use it will be convenient to use the following graphical notation for the previous series



(20)

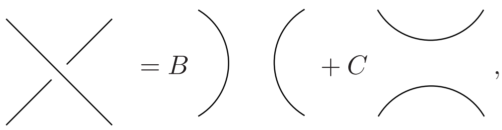
where $z = -io\hbar\lambda$, and the boxes denote symmetrization (18) according to the quantization prescription Q_S defined in (19).

V. SUMMING UP THE PERTURBATIVE SERIES

In this Section we show that the perturbative expansion above can be exactly summed once a definition of the symbol Q is provided. The completely symmetrized ordering $\mathrm{Q} \rightarrow \mathrm{Q}_S$ —which seems natural from the point of view of the Leibniz rule (see remark above)—leads to a complicated result. A different crossing evaluation follows from the action (11) of the flux operator in LQG and the use of the Duflo isomorphism as a quantization map. This possibility, which doesn't seem to contain any physical input but is mathematically preferred, as explained in more detail in the following, leads to the main result (9) of this paper.

A. Symmetric orderings

The symmetric ordering, which we denote Q_S, arises naturally from the above treatment of the path ordered exponentials and the Leibniz rule. As shown in Appendix B, this prescription leads to a closed formula for the crossing, but it doesn't reproduce Kaufmann's bracket algebraic structure; namely, the fully symmetrized ordering yields



(21)

where

$$B(\lambda) = \sin[\hbar\lambda/4](\frac{2i}{3} - \hbar\lambda/4) + \cos[\hbar\lambda/4](1 + \frac{i\hbar\lambda/4}{3})$$

and

$$C(\lambda) = -\sin[\hbar\lambda/4](\frac{2i}{3} + \hbar\lambda/4) + \cos[\hbar\lambda/4](1 - \frac{i\hbar\lambda/4}{3}).$$

One can devise another natural quantization prescription by taking the flux quantization of fluxes of Section III as a guiding principle. Accordingly, there is no quantization ambiguity for the zeroth and first order. At second order the symmetric ordering studied above can be used. As shown Appendix B, the result is proportional to the Casimir E^2. Therefore, the second order term is proportional to the zeroth order. We can define the third order as the result of the (unambiguous) action of a single flux E on the second order. This gives an iterative definition of all orders and produces a quantization prescription that coincides with Q_S up to second order. However, as the previous case also at second order one departures from the Kauffman bracket expected result. We compute for completeness all orders in Appendix B, the result is

$$B(\lambda) = \cos[\sqrt{3}\hbar\lambda/4] - \frac{4i}{\sqrt{3}} \sin[\sqrt{3}\hbar\lambda/4]$$

and

$$C(\lambda) = \cos[\sqrt{3}\hbar\lambda/4] + \frac{4i}{\sqrt{3}} \sin[\sqrt{3}\hbar\lambda/4].$$

This latter quantization prescription, has however, the advantage that all the ambiguities are now confined to the quantization of the Casimir E^2. The key ingredient in the resolution of this remaining ambiguity is the existence of a preferred quantization prescription for Casimirs: the Duflo map.

B. The Duflo map

The Duflo map [22] is a generalization of the universal *quantization map* proposed by Harish-Chandra for semi-simple Lie algebras. The latter provides a prescription to quantize polynomials of commuting variables (the classical triad fields e) which after quantization acquire Lie algebra commutation relations (the flux operators $\hat{E}$). More precisely, given a set of commuting variables E_i on the dual space $\mathfrak{g}^*$ of the algebra $\mathfrak{g}$, they generate the commutative algebra of polynomials, called the *symmetric algebra* over $\mathfrak{g}$ and denoted $\mathrm{Sym}(\mathfrak{g})$. If now we want to map this algebra into the one generated by non-commutative variables τ_i which satisfy the commutation relations $[\tau_i, \tau_j] = f_{ij}{}^k \tau_k$, we run into ordering problem since the commutative algebra $\mathrm{Sym}(\mathfrak{g})$ must be mapped to the non-commutative *universal enveloping algebra* $U(\mathfrak{g})$. A natural quantization map introduced by Harish-Chandra [25] is the so-called symmetric quantization, defined by its action on monomials, namely

$$\mathrm{Q}_S : \; E_{i_1} E_{i_2} \cdots E_{i_n} \quad \to \quad \frac{1}{n!} \sum_{\pi \in S_n} \tau_{i_{\pi(1)}} \tau_{i_{\pi(2)}} \cdots \tau_{i_{\pi(n)}}. \tag{22}$$

A generalization of the previous map was provided by Duflo by composing it with a differential operator $j^{\frac{1}{2}}(\partial)$ on $\mathrm{Sym}(\mathfrak{g})$, where $\partial_i \equiv \partial/\partial E_i$ represents derivatives with respect to the generators of $\mathrm{Sym}(\mathfrak{g})$. In the case of the Lie algebra $\mathfrak{su}(2)$, the Duflo map Q_D reads

$$\mathrm{Q}_D = \mathrm{Q}_S \circ j^{\frac{1}{2}}(\partial) = \mathrm{Q}_S \circ \left(1 + \frac{1}{12} \partial_i \partial_i + \cdots \right), \tag{23}$$

where the dots stand for terms containing higher derivatives.

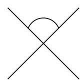
The main property of Q_D is that given two Casimir elements A and B, the product of quantizations $\mathrm{Q}_D(A)\mathrm{Q}_D(B)$ coincides with the quantization of the product, $\mathrm{Q}_D(AB)$. Therefore, the Duflo map is an isomorphism between the invariant (under the action of G) sub-algebras $\mathrm{Sym}(\mathfrak{g})^{\mathfrak{g}}$ and $U(\mathfrak{g})^{\mathfrak{g}}$.

The Duflo map provides a mathematically preferred quantization for products of E; however, such choice is not always physically acceptable. For instance if one would use it for the quantization

of angular momentum in the hydrogen atom one would get an energy spectrum incompatible with observations. In LQG this map has also been proposed to provide an alternative quantization of the area operators [25]. Such choice leads to a simpler area spectrum; however, it has drawback of violating cylindrical consistency [26].

C. Quantization in terms of flux operators

In order to get the general form of the series (20) in the case where we use the quantization of the flux operators given in Section III it suffices to write the first few terms. In the first order term, E acts as a LIV on the portion of the holonomy which has the crossing as its source and as a RIV on the other one. The full result is, just as in (20):

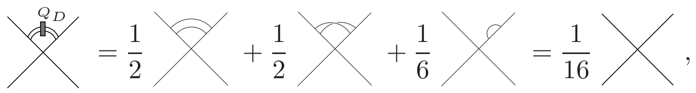


(24)

In the second order diagram we have the action of two flux operators at the same point and therefore ordering ambiguities arise. In order to deal with them, we now use the prescription induced by the Duflo map, namely we write $(\tau_j \tau_k)$ as

$$\begin{aligned} \mathrm{Q}_D[E_j E_k] &= \mathrm{Q}_S \circ \left(1 + \frac{1}{12}\partial_i \partial_i + \cdots \right)[E_j E_k] \\ &= \frac{1}{2}(\tau_j \tau_k + \tau_k \tau_j) + \frac{1}{6}\delta_{jk}. \end{aligned} \tag{25}$$

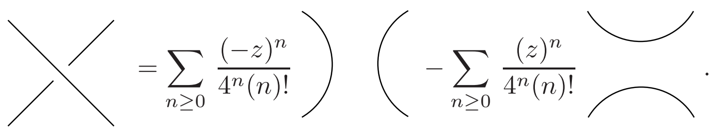
Diagrammatically, for the second order term we have



(26)

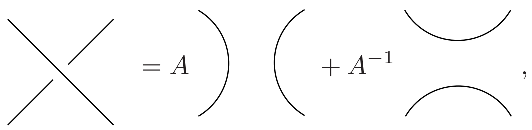
where in the second equality we used the fact that $\{\tau^i, \tau^j\} = -1/2\, \delta^{ij}$ and the value of the Casimir in the fundamental representation. Therefore, the second order diagram is proportional to the order zero diagram. The third order term is consequently proportional to the first order one and so on [6]. We get in this way the general expression for arbitrary order. Finally, choosing an orientation and using equations (A1) and (A2) in the Appendix A, we can express the ordered version of Equation (17) as

$h_\eta (A_\lambda) \triangleright h_\gamma (A_\lambda)\ |0\rangle =$



(27)

Therefore, the series expansion in powers of λ converges and leads to a simple expression for the crossing. Using Penrose convention $\epsilon_{AB} \to i\epsilon_{AB}$ and $\epsilon^{AB} \to i\epsilon^{AB}$ to take care of the different relative signs, the result is



(28)

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[6] Notice that, if we haven't used the quantization scheme of the flux operators proper of the LQG formalism, in order to compute the terms beyond the second order, we should have applied the Duflo map at all orders (i.e. compute the action of Q_D on all the other products of Es). This alternative prescription (besides being much more involved) would lead to a result differing from the reproduction of the Kaufmann bracket, thus showing the central role played by the LQG representation of the fundamental variables.

where $A = e^{\frac{io\hbar\lambda}{4}}$, with o the relative orientation between η and γ. Equations (28) has the same form as Kauffman's q-deformed binor identity for $q = \exp i\lambda/2$.

VI. DISCUSSION

We have shown that the holonomy of A_λ in the fundamental representation can be quantized in different ways due to ordering ambiguities. However, there exists a simple and natural quantization based on the Duflo map leading to the Kauffman-like algebraic structure for the action of the quantum holonomy defining a crossing. This result is promissing in the road to finding a relationship between Turaev-Viro amplitudes and physical amplitudes in canonical LQG.

The recovering of the Kauffman bracket related to the q-deformed crossing identity is a remarkable result since it was obtained starting from the standard $SU(2)$ kinematical Hilbert space of LQG and combining the flux operators representation of the theory together with a mathematical input coming from the Duflo isomorphism. The fact that the crossing of our quantum holonomies have this structure is an encouraging result in finding a link between the role of quantum groups in 3d gravity with non vanishing cosmological constant and its canonical quantization. However, the full link can only be established if the dynamical input from the implementation of the constraints (4) is brought in. Quantum holonomies defined here might be the right tool for regularizing the quantum constraints as proposed in (6).

As pointed out in the previous paragraph and at the end of Section II, the topological features of knots (Reidermeister moves) as well as the related quantum evaluation of Wilson loops is only to be found through dynamical considerations. Since in the present analysis no quantum group structure has been introduced by hand at any stage, at the present kinematical level, loops still evaluate according to the classical $SU(2)$ recoupling theory. Nevertheless, an intriguing indication that the implementation of dynamics could lead to the emergence of the quantum dimension for loops evaluation is available already at this stage. More precisely, if one takes seriously the expression (6) as a proposal for the regularized version of the curvature constraint (4)—notice that, in the naive continuum limit, the expression (4) is recovered—then one could compute it's algebra by studying the action of the commutator on some states. The classical constraint algebra dictates that this should be proportional to the Gauss constraint. If one performs this analysis, it is immediate to see that there are two types of anomalous contributions: one of the same kind of the anomaly found in [20] (which could be called mild as the terms produced vanish when acting on gauge invariant states), and another anomalous contribution (a stronger one) that does not annihilate gauge invariant states. The latter anomalous terms happen to be proportional ($A^2 + A^{-2} +$ ), where  represents the loop with no area in the fundamental representation $j = 1/2$. Thus the condition that an infinitesimal loop evaluates to the quantum dimension $-A^2 - A^{-2}$ emerges from the constraint algebra: the anomaly is proportional to the difference of the quantum and classical evaluation of the loop.

All this indicates that, even when we do not introduce a quantum group at any stage, and no dynamical constraint has been imposed yet, amplitudes such as the value of the quantum dimension (or self linking number of a Wilson loop in the language of [1]) $d_q = -q - q^{-1}$ and $q = A^2 = \exp i\hbar\lambda/4$ naturally appear from our treatment. Recall that the value of d_q together with the deformed binor identity are the two ingredients for the combinatorial definition of the Turaev-Viro invariant according to the formulation of [16]. This is encouraging as it indicates that perhaps a strict correspondence between LQG and the Tuarev-Viro invariant can be established if one appropriately implements the next step: quantizing and imposing the curvature constraint (4). This will be investigated in the future.

An interesting correspondence between operator ordering and time was found in [8] (see also [17]). This relationship is expected to be more explicit here. Notice that even though the canonical quantization is defined on the 2-dimensional manifold Σ, the non commutativity of the quantum holonomy, can be encoded in terms of the knotting of paths as if they would be embedded in a 3-dimensional manifold of topology $\Sigma \times \mathbb{R}$. If the quantum constraints can be imposed as in the zero

cosmological constant case, we expect the expectation value of these knots in the physical Hilbert space to coincide with the ones computed using the covariant methods of [1]. This would be an explicit example where operators defined in the 'frozen' timeless formalism of Dirac can be directly interpreted as space-time processes. Such an example would be of great conceptual importance showing that the notion of time and causality can be encoded in the quantum theory defined on a single space slice.

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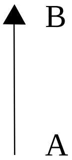
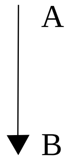
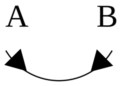
Appendix A: Diagrammatic algebra

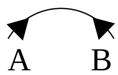
Many results connected to the theory of representation of $SU(2)$ can be more easily stated in a graphical notation introduced by Penrose. The association of an algebraic meaning to the various diagrams is subject to many conventions; therefore, here we present ours.

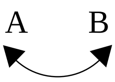
To every single arrow 

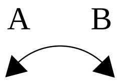
or 

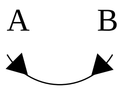
going from index A to index B associate the symbol δ^A_B (note that it does not matter whether the arrow is up- or down-going).

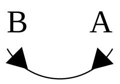
To every symbol 

or 

(ingoing arrows) associate the object ϵ^{AB} and to every symbol 

or 

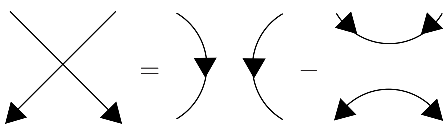
(outgoing arrows) the object ϵ_{AB}, where $(\epsilon_{AB}) = (\epsilon^{AB}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (note that it does not matter whether the arc is convex or concave). Note also that since ϵ is antisymmetric, 

is $-$ 

.

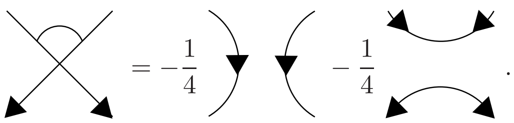
It is also important to note that it does not matter whether the strands are vertical or horizontal, the only important thing being the direction of the arrows and the reading order of the indices.

With these conventions, it is easy to check that (Penrose's "binor identity" for $SU(2)$)



(A1)

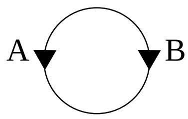
and that

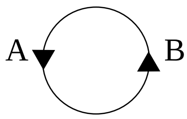


(A2)

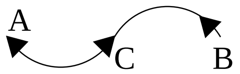
It is enough to rotate these diagrams in order to get the identities corresponding to the other three possible choices of arrows.

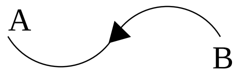
We also have that



$$= \epsilon_{AB}\epsilon^{AB} = 2 = \delta^A_B \delta^B_A =$$


and that



$$= \epsilon_{AC}\epsilon^{CB} = -\delta^B_A = -$$


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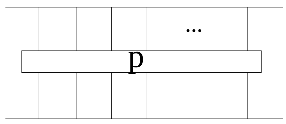
Appendix B: Symmetric ordering

Here we explore the quantization of the quantum holonomy based on the symmetrized 'factor ordering' at the level of (20). This amounts to replacing the term $\tau_{(i_{k_1}} \cdots \tau_{i_{k_p})}$ in Equation (17) by $\mathrm{Q}_S(E_{i_{k_1}} \cdots E_{i_{k_p}}) = \frac{1}{p!}\sum_{\pi \in S(p)} \tau_{i_{\pi(1)}}\tau_{i_{\pi(2)}} \cdots \tau_{i_{\pi(p)}}$.

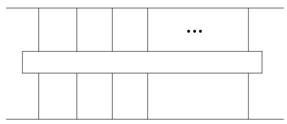
We introduce the Penrose graphical notation

$$\tau^{i_1} \cdots \tau^{i_p} \otimes \tau_{(i_1} \cdots \tau_{i_p)} =$$


,

where the vertical lines represent the contraction of the i-indices, the 3-valent nodes denote the τ-matrices, the horizontal lines represent the contraction of the spinor indices, i.e., matrix product, and the box in the middle denotes the symmetrization of the i-indices.

Using the fact that $\{\tau^i, \tau^j\} = -2\delta^{ij}$ it is immediate to proove the following identities:



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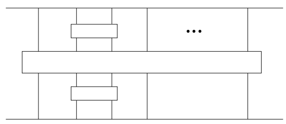

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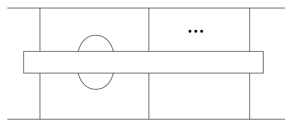
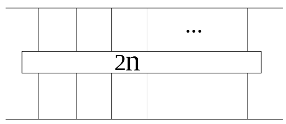


(B1)

which imply



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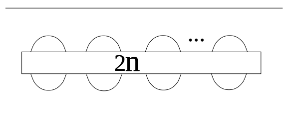
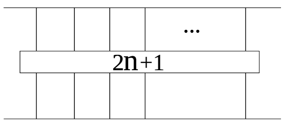


$= A_{2n}$

, (B2)

where in the last equality we have introduced the definition of the coefficient A_{2n}, and



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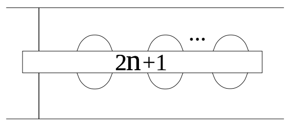
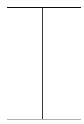


$= B_{2n+1}$

, (B3)

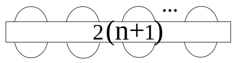
where in the last equality we use the fact the the diagram between the horizontal lines is proportional to the identity in order to introduce the definition of the coefficient B_{2n+1}. Indeed the previous equations can be written in the standard tensorial notation as:

$$\tau^{i_1} \cdots \tau^{i_{2n}} \otimes \tau_{(i_1} \cdots \tau_{i_{2n})} = A_{2n}(1 \otimes 1), \tag{B4}$$

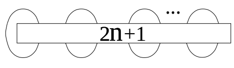
and

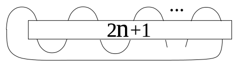
$$\tau^{i_1} \cdots \tau^{i_{2n+1}} \otimes \tau_{(i_1} \cdots \tau_{i_{2n+1})} = B_{2n+1} \left(\tau^i \otimes \tau_i\right) \tag{B5}$$

In order to compute the coefficients A_{2n} and B_{2n+1} we observe that



$=$

$\frac{1}{2n+2}\Bigg($ 

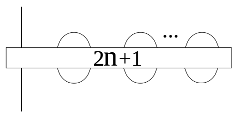
$+$ 

$+ \text{cyclic permutations}\Bigg)$, (B6)

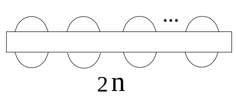
which is a simple property of symmetric tensors. But each term on the righthand side is equal to B_{2n+1} times the trace of the identity (see equation B3) Therefore, we have proven that

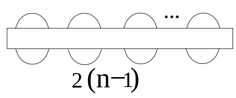
$$3B_{2n} = A_{2(n+1)}. \tag{B7}$$

The is also a simple recursion relation relating the unknown coefficients which diagrammatically takes the following form:



$=$

$\frac{N_0}{2n+1}$ 

$+ \frac{N_1}{(2n+1)(2n)(2n-1)}$ 

$+ \cdots + N_n \Big|$,

where the factors N_j for $0 \le j \le n$ correspond to the number of ways one can start at the top vertical line go around the symmetrization box and exit along the bottom vertical line by 'walking' along j upper and j bottom arcs respectively. It is easy to see that $N_0 = 1$, $N_1 = (2n)^2$ (after entering the box we have $2n$ choices to enter one of the arcs in the bottom times $2n$ choices on the top) the general term being

$$N_j = [(2n)(2(n-1)) \cdots (2(n-j))]^2 = 2^{2j} \left[\frac{n!}{(n-j)!}\right]^2$$

The other explicit coefficients in front of each term just come from the readjustment of the number of permutations. For instance in the first term $1/(2n+1)$ times the $1/(2n)!$ gives corresponding to the symmetrization factor on the left $1/(2n+1)!$. Similarly for the second term we have $1/((2n+1)(2n)(2n-1))$ times $1/(2(n-1))!$ gives again $1/(2n+1)!$. The general term being $(2(n-j))!/(2n+1)!$. Putting all this together we get

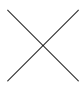
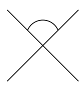
$$B_{2n+1} = \sum_{j=0}^{n} 2^{2j} \frac{[2(n-j)]!}{(2n+1)!} \left[\frac{n!}{(n-j)!}\right]^2 A_{2(n-j)} \tag{B8}$$

combining the two equations the solution is:

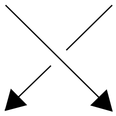
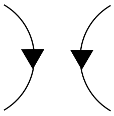
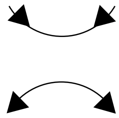
$$A_{2n} = 2n+1 \qquad B_{2n+1} = \frac{2}{3}n + 1 \tag{B9}$$

With this result the symmetrized version of Equation (17) yields

$$\sum_{n\ge0} \frac{(-\mathrm{i}o\hbar\lambda/4)^p}{p!} \tau^{i_{k_1}} \cdots \tau^{i_{k_p}} \otimes \tau_{(i_{k_1}} \cdots \tau_{i_{k_p})} =$$

$= \sum_{n\ge0} \frac{(-\mathrm{i}o\hbar\lambda/4)^{2n}}{(2n)!}(2n+1)$  $+ \sum_{n\ge0} \frac{(-\mathrm{i}o\hbar\lambda/4)^{2n+1}}{(2n+1)!}\left(\frac{2}{3}n+1\right)$ 

Finally, choosing an orientation and using eq. (A1)-(A2) we arrive at the result

 $= B$  $- C$  , (B10)

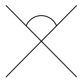
where

$$B(\lambda) = \sin[\hbar\lambda/4](\frac{2i}{3} - \hbar\lambda/4) + \cos[\hbar\lambda/4](1 + \frac{i\hbar\lambda/4}{3})$$

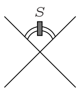
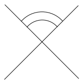
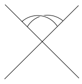
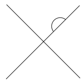
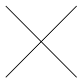
and

$$C(\lambda) = -\sin[\hbar\lambda/4](\frac{2i}{3} + \hbar\lambda/4) + \cos[\hbar\lambda/4](1 - \frac{i\hbar\lambda/4}{3}).$$

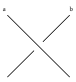
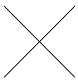
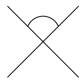
Therefore, considering the totally symmetric map Q_S leads to the wrong result. Another possibility consists of trying to improve this map taking into account the action of the flux operators. More precisely, along the lines of Section V C, the unambiguous first order term is again given by

 . (B11)

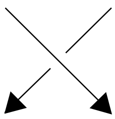
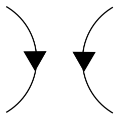
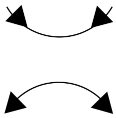
Then, the second order diagram can be viewed as the result on an action of the flux operator on the first order diagram. We now apply the symmetrization map Q_S to compute this action, namely

 $= \frac{1}{2}$  $+ \frac{1}{2}$  $= -\frac{1}{4}$  $= \frac{3}{16}$  , (B12)

where in the first equation we get two terms coming from on LIV action and a RIV action, while in the second equality we use the fact that $\{\tau^i, \tau^j\} = -1/2\ \delta^{ij}$. Therefore, the second order diagram is proportional to the order zero diagram. The proportionality constant is just 1/4 of the value of the Casimir in the fundamental representation. The third order term is consequently proportional to the first order one and so on. We get in this way the general expression for arbitrary order. With this prescription the result of the quantum holonomy action now becomes

 $= \sum_{n \geq 0} \frac{(-\mathrm{i}o\hbar\lambda)^{2n}}{(2n)!} \left(\frac{3}{16}\right)^n$  $+ \sum_{n \geq 0} \frac{(-\mathrm{i}o\hbar\lambda)^{2n+1}}{(2n+1)!} \left(\frac{3}{16}\right)^n$  ,

which again, through eq. (A1)-(A2), can be written as

 $= B$  $- C$  , (B13)

where

$$B(\lambda) = \cos[\sqrt{3}\hbar\lambda/4] - \frac{4i}{\sqrt{3}} \sin[\sqrt{3}\hbar\lambda/4]$$

and

$$C(\lambda) = \cos[\sqrt{3}\hbar\lambda/4] + \frac{4i}{\sqrt{3}} \sin[\sqrt{3}\hbar\lambda/4].$$

The results (B10)-(B13) show how, using some 'first guess' ordering to solve the multiple flux operators action ambiguity, one can obtain a series expansion in powers of λ which converges and leads to a simple expression for the crossing, but doesn't reproduce the expected result.

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